

The Succinctness of First-order Logic over Modal Logic via a Formula Size Game

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Abstract

We propose a new version of formula size game for modal logic. The game characterizes the equivalence of pointed Kripke-models up to formulas of given numbers of modal operators and binary connectives. Our game is similar to the well-known Adler-Immerman game. However, due to a crucial difference in the definition of positions of the game, its winning condition is simpler, and the second player (duplicator) does not have a trivial optimal strategy. Thus, unlike the Adler-Immerman game, our game is a genuine two-person game. We illustrate the use of the game by proving a nonelementary succinctness gap between bisimulation invariant first-order logic FO and (basic) modal logic ML.

Keywords: Succinctness, formula size game, bisimulation invariant first-order logic, n -bisimulation.

1 Introduction

Succinctness is an important research topic that has been quite active in modal logic for the last couple of decades; see, e.g., [3,13,11,14,1,12,5] for earlier work on this topic and [6,18,7,10,17,19] for recent research. If two logics \mathcal{L} and \mathcal{L}' have equal expressive power, it is natural to ask, whether there are properties that can be expressed in \mathcal{L} by a substantially shorter formula than in \mathcal{L}' (or vice versa). For example, \mathcal{L} is *exponentially more succinct* than \mathcal{L}' , if for every integer n there is an \mathcal{L} -formula φ_n of length $\mathcal{O}(n)$ such that any equivalent \mathcal{L}' -formula ψ_n is of length at least 2^n .

Often such a gap in succinctness comes together with a similar gap in the complexity of the logics. For example, Etessami, Vardi and Wilke [3] proved that, over ω -words, the two-variable fragment FO^2 of first-order logic has the same expressive power as unary-TL (a weak version of temporal logic), but FO^2 is exponentially more succinct than unary-TL, and furthermore, the complexity of satisfiability for FO^2 is NEXPTIME-complete, while the complexity of unary-TL is in NP [15]. However, succinctness does not always lead to a penalty in terms of complexity: an example is public announcement logic PAL which is exponentially more succinct than epistemic logic EL, but both have the same complexity, as proved by Lutz in [12].

In order to prove succinctness results we need a method for proving lower bounds for the length of formulas expressing given properties. The two most common methods used in the recent literature are the *formula size game* introduced by Adler and Immerman [1], and *extended syntax trees* due to Grohe and Schweikardt [8]. The latter was inspired by the former, and in fact, an extended syntax tree is essentially a witness for the existence of a winning strategy in the Adler-Immerman game. Thus, these two methods are equivalent, and the choice between them is often a matter of convenience.

Originally, Adler and Immerman [1] formulated their game for the branching-time temporal logic CTL. They used it for proving an $n!$ lower bound on the size of CTL-formulas for expressing that there is a path on which each of the propositions p_1, \dots, p_n is true. As it is straightforward to express this property by a formula of CTL^+ of size linear in n , their result established that CTL^+ is $n!$ times more succinct than CTL, thus improving an earlier exponential succinctness result of Wilke [20].

After its introduction in [1], the Adler-Immerman game, as well as the method of extended syntax trees, has been adapted to a host of modal languages. These include epistemic logic [6], multimodal logics with union and intersection operators on modalities [18] and modal logic with contingency operator [19], among others.

The Adler-Immerman game can be seen as a variation of the Ehrenfeucht-Fraïssé game, or, in the case of modal logics, the bisimulation game. In the Adler-Immerman game, quantifier rank (or modal depth) is replaced by a parameter, usually called formula size, that is closely related to the length of the formula. Moreover, in order to use the game for proving that a property is not definable by a formula of a given size, it is necessary to play the game on pair (\mathbb{A}, \mathbb{B}) of sets of structures instead of just a pair of single structures.

The basic idea of the Adler-Immerman game is that one of the players, S (spoiler), tries to show that the sets \mathbb{A} and \mathbb{B} can be separated by a formula of size n , while the other player, D (duplicator), aims to show that no formula of size at most n suffices for this. The moves that S makes in the game reflect directly the logical operators in a formula that is supposed to separate the sets \mathbb{A} and \mathbb{B} . Any pair (σ, δ) of strategies for the players S and D produces a finite game tree $T_{\sigma, \delta}$, and S wins this play if the size of $T_{\sigma, \delta}$ is at most n . The strategy σ is a winning strategy for S if using it, S wins every play of the game. If this is the case, then there is a formula of size at most n that separates the sets, and this formula can actually be read from the strategy σ .

A peculiar feature of the Adler-Immerman game is that the second player, duplicator, can be completely eliminated from it. This is because D has an optimal strategy δ_{\max} , which is to always choose the maximal allowed answer; this strategy guarantees that the size of the tree $T_{\sigma, \delta}$ is as large as possible. Thus, in this sense the Adler-Immerman game is not a genuine two-person game, but rather a one-person game.

In the present paper, we propose another type of formula size game for modal logic. Our game is a natural adaptation of the game introduced by

Hella and Väänänen [9] for propositional logic and first-order logic. The basic setting in our game is the same as in the Adler-Immerman game: there are two players, S and D, and two sets of structures that S claims can be separated by a formula of some given size. The crucial difference is that in our game we define positions to be tuples $(m, k, \mathbb{A}, \mathbb{B})$ instead of just pairs (\mathbb{A}, \mathbb{B}) of sets of structures, where m and k are parameters referring to the number of modal operators and binary connectives in a formula. In each move S has to decrease at least one of the parameters m or k . The game ends when the players reach a position $(m^*, k^*, \mathbb{A}^*, \mathbb{B}^*)$ such that either there is a literal separating \mathbb{A}^* and \mathbb{B}^* , or $m^* = k^* = 0$. In the former case, S wins the play; otherwise D wins.

Thus, in contrast to the Adler-Immerman game, to determine the winner in our game it suffices to consider a single “leaf-node” $(m^*, k^*, \mathbb{A}^*, \mathbb{B}^*)$ of the game tree. This also means that our game is a real two-person game: the final position $(m^*, k^*, \mathbb{A}^*, \mathbb{B}^*)$ of a play depends on the moves of D, and there is no simple optimal strategy for D that could be used for eliminating the role of D in the game.

We believe that our game is more intuitive and thus, in some cases it may be easier to use than the Adler-Immerman game. On the other hand, it should be remarked that the two games are essentially equivalent: The moves corresponding to connectives and modal operators are the same in both games (when restricting to the sets \mathbb{A} and \mathbb{B} in a position $(m, k, \mathbb{A}, \mathbb{B})$). Hence, in principle, it is possible to translate a winning strategy in one of the games to a corresponding winning strategy in the other.

We illustrate the use of our game by proving a nonelementary succinctness gap between first-order logic FO and (basic) modal logic ML. More precisely, we define a bisimulation invariant property of pointed Kripke models by a first-order formula of size $\mathcal{O}(2^n)$, and show that this property cannot be defined by any ML-formula of size less than the exponential tower of height $n - 1$.

A similar gap between FO and temporal logic follows from a construction in the PhD thesis [16] of Stockmeyer. He proved that the satisfiability problem of FO over words is of nonelementary complexity. Etessami and Wilke [4] observed that from Stockmeyer’s proof it is possible to extract FO-formulas of size $\mathcal{O}(n)$ whose smallest models are words of length nonelementary in n . On the other hand, it is well known that any satisfiable formula of temporal logic has a model of size $\mathcal{O}(2^n)$, where n is the size of the formula.

2 Preliminaries

In this section we fix some notation, define the syntax and semantics of basic modal logic and define our notions of formula size. For a detailed account on the notions used in the paper, we refer to the textbook [2] of Blackburn, de Rijke and Venema.

Basic modal logic and first-order logic

Let $\mathcal{M} = (W, R, V)$, where W is a set, $R \subseteq W \times W$ and $V : \Phi \rightarrow \mathcal{P}(W)$, and let $w \in W$. The structure (\mathcal{M}, w) is called a *pointed Kripke-model* for Φ .

Let (\mathcal{M}, w) be a pointed Kripke-model. We use the notation

$$\Box(\mathcal{M}, w) := \{(\mathcal{M}, v) \mid v \in M, wR^{\mathcal{M}}v\}.$$

If \mathbb{A} is a set of pointed Kripke-models, we use the notation

$$\Box\mathbb{A} := \bigcup_{(\mathcal{M}, w) \in \mathbb{A}} \Box(\mathcal{M}, w).$$

Furthermore, if f is a function $f : \mathbb{A} \rightarrow \Box\mathbb{A}$ such that $f(\mathcal{M}, w) \in \Box(\mathcal{M}, w)$ for every $(\mathcal{M}, w) \in \mathbb{A}$, then we use the notation

$$\Diamond_f \mathbb{A} := f(\mathbb{A}).$$

Now we define the syntax and semantics of basic modal logic for pointed models.

Definition 2.1 Let Φ be a set of proposition symbols. The set of formulas of $\text{ML}(\Phi)$ is generated by the following grammar

$$\varphi := p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond \varphi \mid \Box \varphi,$$

where $p \in \Phi$.

As is apparent from the definition of the syntax, we assume that all ML-formulas are in negation normal form. This is useful for the formula size game that we introduce in the next section.

Definition 2.2 The satisfaction relation $(\mathcal{M}, w) \models \varphi$ between pointed Kripke-models (\mathcal{M}, w) ML(Φ)-formulas φ is defined as follows:

- (1) $(\mathcal{M}, w) \models p \Leftrightarrow w \in V(p)$,
- (2) $(\mathcal{M}, w) \models \neg p \Leftrightarrow w \notin V(p)$,
- (3) $(\mathcal{M}, w) \models (\varphi \wedge \psi) \Leftrightarrow (\mathcal{M}, w) \models \varphi$ and $(\mathcal{M}, w) \models \psi$,
- (4) $(\mathcal{M}, w) \models (\varphi \vee \psi) \Leftrightarrow (\mathcal{M}, w) \models \varphi$ or $(\mathcal{M}, w) \models \psi$,
- (5) $(\mathcal{M}, w) \models \Diamond \varphi \Leftrightarrow$ there is $(\mathcal{M}, v) \in \Box(\mathcal{M}, w)$ such that $(\mathcal{M}, v) \models \varphi$,
- (6) $(\mathcal{M}, w) \models \Box \varphi \Leftrightarrow$ for every $(\mathcal{M}, v) \in \Box(\mathcal{M}, w)$ it holds that $(\mathcal{M}, v) \models \varphi$.

Furthermore, if \mathbb{A} is a class of pointed Kripke-models, then

$$\mathbb{A} \models \varphi \Leftrightarrow (\mathcal{A}, w) \models \varphi \text{ for every } (\mathcal{A}, w) \in \mathbb{A}.$$

In Section 4, we also consider the case $\Phi = \emptyset$. For this purpose, we add the atomic constants \top and \perp to ML, where $(\mathcal{M}, w) \models \top$ and $(\mathcal{M}, w) \not\models \perp$ for all pointed Kripke models (\mathcal{M}, w) .

The syntax and semantics for first-order logic are defined in the standard way. Each ML-formula φ defines a class $\text{Mod}(\varphi)$ of pointed Kripke-models:

$$\text{Mod}(\varphi) := \{(\mathcal{M}, w) \mid (\mathcal{M}, w) \models \varphi\}.$$

In the same way, any FO-formula $\psi(x)$ in the vocabulary consisting of the accessibility relation symbol R and unary relation symbols U_p for $p \in \Phi$ defines a class $\text{Mod}(\psi)$ of pointed Kripke-models:

$$\text{Mod}(\psi) := \{(\mathcal{M}, w) \mid \mathcal{M} \models \psi[w/x]\}.$$

The formulas $\varphi \in \text{ML}$ and $\psi(x) \in \text{FO}$ are *equivalent* if $\text{Mod}(\varphi) = \text{Mod}(\psi)$.

The well-known link between ML and FO is the following theorem.

Theorem 2.3 (van Benthem Characterization Theorem) *A first-order formula $\psi(x)$ is equivalent to some formula in ML if and only if $\text{Mod}(\psi)$ is bisimulation invariant.*

If a property of pointed Kripke-models is n -bisimulation invariant for some $n \in \mathbb{N}$, then it is also bisimulation invariant. Thus, FO-definability and n -bisimulation invariance imply ML-definability for any property of pointed Kripke-models. We will use this version of van Benthem's characterization in Section 4.1 for showing that certain property is ML-definable. For the sake of easier reading, we give here the definition of n -bisimulation.

Definition 2.4 Let (\mathcal{M}, w) and (\mathcal{M}', w') be pointed models. We say that (\mathcal{M}, w) and (\mathcal{M}', w') are n -bisimilar, $(\mathcal{M}, w) \simeq_n (\mathcal{M}', w')$, if there are binary relations $Z_n \subseteq \dots \subseteq Z_0$ such that for every $0 \leq i \leq n-1$ we have

- (1) $(\mathcal{M}, w)Z_n(\mathcal{M}', w')$,
- (2) if $(\mathcal{M}, v)Z_0(\mathcal{M}', v')$, then v and v' are propositionally equivalent,
- (3) if $(\mathcal{M}, v)Z_{i+1}(\mathcal{M}', v')$ and $(\mathcal{M}, u) \in \Box(\mathcal{M}, v)$ then there is $(\mathcal{M}', u') \in \Box(\mathcal{M}', v')$ such that $(\mathcal{M}, u)Z_i(\mathcal{M}', u')$,
- (4) if $(\mathcal{M}, v)Z_{i+1}(\mathcal{M}', v')$ and $(\mathcal{M}', u') \in \Box(\mathcal{M}', v')$ then there is $(\mathcal{M}, u) \in \Box(\mathcal{M}, v)$ such that $(\mathcal{M}, u)Z_i(\mathcal{M}', u')$.

It is well known that two pointed Kripke-models are n -bisimilar if and only if they are equivalent with respect to ML-formulas of modal depth at most n .

Formula size

We define notions of formula size for ML and FO. These notions are related to the length of the formula as a string rather than the DAG-size¹ of it. For ML we define separately the number of modal operators and the number of binary connectives in the formula.

Definition 2.5 The *modal size* of a formula $\varphi \in \text{ML}$, denoted $\text{ms}(\varphi)$, is defined recursively as follows:

- (1) If φ is a literal, then $\text{ms}(\varphi) = 0$.
- (2) If $\varphi = \psi \vee \vartheta$ or $\varphi = \psi \wedge \vartheta$, then $\text{ms}(\varphi) = \text{ms}(\psi) + \text{ms}(\vartheta)$.
- (3) If $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$, then $\text{ms}(\varphi) = \text{ms}(\psi) + 1$.

¹ The DAG-size of a formula φ is the size of the syntactic structure of φ in the form of a DAG. Thus, it is less than n^2 , where n is the number of subformulas of φ .

Definition 2.6 The *binary connective size* of a formula $\varphi \in \text{ML}$, denoted by $\text{cs}(\varphi)$, is defined recursively as follows:

- (1) If φ is a literal, then $\text{cs}(\varphi) = 0$.
- (2) If $\varphi = \psi \vee \vartheta$ or $\varphi = \psi \wedge \vartheta$, then $\text{cs}(\varphi) = \text{cs}(\psi) + \text{cs}(\vartheta) + 1$.
- (3) If $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$, then $\text{cs}(\varphi) = \text{cs}(\psi)$.

The size of an ML formula is defined as the sum of modal size and connective size. We do not count literals or parentheses since their number can be derived from the number of binary connectives.

Definition 2.7 The *size* of a formula $\varphi \in \text{ML}$ is $s(\varphi) = \text{ms}(\varphi) + \text{cs}(\varphi)$.

Similarly we define formula size for FO to be the number of binary connectives and quantifiers in the formula. In general this could lead to an arbitrarily large difference between formula size and actual string length, but we only consider formulas with one binary relation so this is not an issue.

Definition 2.8 The *size* of a formula $\varphi \in \text{FO}$, denoted by $s(\varphi)$, is defined recursively as follows:

- (1) If φ is a literal, then $s(\varphi) = 0$.
- (2) If $\varphi = \neg\psi$, then $s(\varphi) = s(\psi)$.
- (3) If $\varphi = \psi \vee \vartheta$ or $\varphi = \psi \wedge \vartheta$, then $s(\varphi) = s(\psi) + s(\vartheta) + 1$.
- (4) If $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$, then $s(\varphi) = s(\psi) + 1$.

To refer to some rather large formula sizes we need the exponential tower function.

Definition 2.9 We define the function $\text{twr} : \mathbb{N} \rightarrow \mathbb{N}$ recursively as follows:

$$\begin{aligned} \text{twr}(0) &= 1 \\ \text{twr}(n+1) &= 2^{\text{twr}(n)}. \end{aligned}$$

We will also use in the sequel the binary logarithm function, denoted by \log .

Separating classes by formulas

The definition of the formula size game in the next section is based on the notion of separating classes of pointed Kripke-models by formulas.

Definition 2.10 Let \mathbb{A} and \mathbb{B} be classes of pointed Kripke-models.

- (a) We say that a formula $\varphi \in \text{ML}$ *separates the classes* \mathbb{A} and \mathbb{B} if $\mathbb{A} \models \varphi$ and $\mathbb{B} \models \neg\varphi$.
- (b) Similarly, a formula $\psi(x) \in \text{FO}$ separates the classes \mathbb{A} and \mathbb{B} if for all $(\mathcal{M}, w) \in \mathbb{A}$, $\mathcal{M} \models \psi[w/x]$ and for all $(\mathcal{M}, w) \in \mathbb{B}$, $\mathcal{M} \models \neg\psi[w/x]$.

In other words, a formula $\varphi \in \text{ML}$ separates the classes \mathbb{A} and \mathbb{B} if $\mathbb{A} \subseteq \text{Mod}(\varphi)$ and $\mathbb{B} \subseteq \overline{\text{Mod}(\varphi)}$, where $\overline{\text{Mod}(\varphi)}$ is the complement of $\text{Mod}(\varphi)$.

3 The formula size game

As in the Adler-Immerman game, the basic idea in our formula size game is that there are two players, S (spoiler) and D (duplicator), who play on a pair (\mathbb{A}, \mathbb{B}) of two sets of pointed Kripke-models. The aim of S is to show that \mathbb{A} and \mathbb{B} can be separated by a formula with modal size at most m and connective size at most k , while D tries to refute this. The moves of S reflect the connectives and modal operators of a formula that is supposed to separate the sets.

The crucial difference between our game and the Adler-Immerman game is that we define positions in the game to be tuples $(m, k, \mathbb{A}, \mathbb{B})$ instead of just pairs (\mathbb{A}, \mathbb{B}) . This means that in the connective moves, D has a genuine choice to make. Furthermore, the winning condition of the game is based on a natural property of single positions instead of the size of the entire game tree.

We give now the precise definition of our game.

Definition 3.1 Let \mathbb{A}_0 and \mathbb{B}_0 be sets of pointed Kripke-models and let $m_0, k_0 \in \mathbb{N}$. The formula size game between the sets \mathbb{A}_0 and \mathbb{B}_0 , denoted $\text{FS}_{m_0, k_0}(\mathbb{A}_0, \mathbb{B}_0)$, has two players, S and D. The number m_0 is the *modal parameter* and k_0 is the *connective parameter* of the game. The starting position of the game is $(m_0, k_0, \mathbb{A}_0, \mathbb{B}_0)$. Let the position after n moves be $(m, k, \mathbb{A}, \mathbb{B})$. To continue the game, S has the following four moves to choose from:

- *Left splitting move*: First, S chooses natural numbers m_1, m_2, k_1 and k_2 and sets \mathbb{A}_1 and \mathbb{A}_2 such that $m_1 + m_2 = m$, $k_1 + k_2 + 1 = k$ and $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. Then D decides whether the game continues from the position $(m_1, k_1, \mathbb{A}_1, \mathbb{B})$ or the position $(m_2, k_2, \mathbb{A}_2, \mathbb{B})$.
- *Right splitting move*: First, S chooses natural numbers m_1, m_2, k_1 and k_2 and sets \mathbb{B}_1 and \mathbb{B}_2 such that $m_1 + m_2 = m$, $k_1 + k_2 + 1 = k$ and $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}$. Then D decides whether the game continues from the position $(m_1, k_1, \mathbb{A}, \mathbb{B}_1)$ or the position $(m_2, k_2, \mathbb{A}, \mathbb{B}_2)$.
- *Left successor move*: S chooses a function $f : \mathbb{A} \rightarrow \Box \mathbb{A}$ such that $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ for all $(\mathcal{A}, w) \in \mathbb{A}$ and the game continues from the position $(m - 1, k, \Diamond_f \mathbb{A}, \Box \mathbb{B})$.
- *Right successor move*: S chooses a function $g : \mathbb{B} \rightarrow \Box \mathbb{B}$ such that $g(\mathcal{B}, w) \in \Box(\mathcal{B}, w)$ for all $(\mathcal{B}, w) \in \mathbb{B}$ and the game continues from the position $(m - 1, k, \Box \mathbb{A}, \Diamond_g \mathbb{B})$.

The game ends and S wins in a position $(m, k, \mathbb{A}, \mathbb{B})$ if there is a literal φ such that φ separates the sets \mathbb{A} and \mathbb{B} . The game ends and D wins in a position $(m, k, \mathbb{A}, \mathbb{B})$ if $m = k = 0$ and S does not win in this position.

Note that if $\Box(\mathcal{M}, w) = \emptyset$ for some $(\mathcal{M}, w) \in \mathbb{A}$ ($\in \mathbb{B}$) then S cannot make a left (right) successor move in the position $(m, k, \mathbb{A}, \mathbb{B})$.

We prove now that the formula size game indeed characterizes the separation of two sets of pointed Kripke-models by a formula of a given size.

Theorem 3.2 Let \mathbb{A} and \mathbb{B} be sets of pointed models and let m and k be natural numbers. Then the following conditions are equivalent:

$(\text{win})_{m,k}$ *S has a winning strategy in the game $\text{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.*

$(\text{sep})_{m,k}$ *There is a formula $\varphi \in \text{ML}$ such that $\text{ms}(\varphi) \leq m$, $\text{cs}(\varphi) \leq k$ and the formula φ separates the sets \mathbb{A} and \mathbb{B} .*

Proof. The proof proceeds by induction on the number $m+k$. If $m+k = 0$, no moves can be made. Thus if S wins, then there is a literal φ that separates the sets \mathbb{A} and \mathbb{B} . In this case $s(\varphi) = 0$ so $(\text{win})_{0,0} \Rightarrow (\text{sep})_{0,0}$. On the other hand, if there is a formula φ such that $s(\varphi) \leq 0$ and φ separates the sets \mathbb{A} and \mathbb{B} , then φ is a literal. Thus S wins the game, and we see that $(\text{sep})_{0,0} \Rightarrow (\text{win})_{0,0}$.

Suppose then that $m+k > 0$ and $(\text{win})_{n,l} \Leftrightarrow (\text{sep})_{n,l}$ for all $n, l \in \mathbb{Z}_+$ such that $n+l < m+k$. Assume first that $(\text{win})_{m,k}$ holds. Consider the following cases according to the first move in the winning strategy of S.

- (a) Assume that the first move of the winning strategy of S is a left splitting move choosing numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$ and $k_1 + k_2 + 1 = k$, and sets $\mathbb{A}_1, \mathbb{A}_2 \subseteq \mathbb{A}$ such that $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. Since this move is given by a winning strategy, S has a winning strategy for both possible continuations of the game, $(m_1, k_1, \mathbb{A}_1, \mathbb{B})$ and $(m_2, k_2, \mathbb{A}_2, \mathbb{B})$. Since $m_i + k_i < m_i + k_i + 1 \leq m+k$ for $i \in \{1, 2\}$, by induction hypothesis there is a formula ψ such that $\text{ms}(\psi) \leq m_1$, $\text{cs}(\psi) \leq k_1$ and ψ separates the sets \mathbb{A}_1 and \mathbb{B} and a formula ϑ such that $\text{ms}(\vartheta) \leq m_2$, $\text{cs}(\vartheta) \leq k_2$ and ϑ separates the sets \mathbb{A}_2 and \mathbb{B} . Thus $\mathbb{A}_1 \models \psi$ and $\mathbb{A}_2 \models \vartheta$ so $\mathbb{A} \models \psi \vee \vartheta$. On the other hand $\mathbb{B} \models \neg\psi$ and $\mathbb{B} \models \neg\vartheta$ so $\mathbb{B} \models \neg(\psi \vee \vartheta)$. Therefore the formula $\psi \vee \vartheta$ separates the sets \mathbb{A} and \mathbb{B} . In addition $\text{ms}(\psi \vee \vartheta) = \text{ms}(\psi) + \text{ms}(\vartheta) \leq m_1 + m_2 = m$ and $\text{cs}(\psi \vee \vartheta) = \text{cs}(\psi) + \text{cs}(\vartheta) + 1 \leq k_1 + k_2 + 1 = k$ so $(\text{sep})_{m,k}$ holds.
- (b) The case in which the first move of the winning strategy of S is a right splitting move is proved in the same way as the previous one, with the roles of \mathbb{A} and \mathbb{B} switched, and disjunction replaced by conjunction.
- (c) Assume that the first move of the winning strategy of S is a left successor move choosing a function $f : \mathbb{A} \rightarrow \Box\mathbb{A}$ such that $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ for all $(\mathcal{A}, w) \in \mathbb{A}$. The game continues from the position $(m-1, k, \Diamond_f\mathbb{A}, \Box\mathbb{B})$ and S has a winning strategy from this position. By induction hypothesis there is a formula ψ such that $\text{ms}(\psi) \leq m-1$, $\text{cs}(\psi) \leq k$ and ψ separates the sets $\Diamond_f\mathbb{A}$ and $\Box\mathbb{B}$. Now for every $(\mathcal{A}, w) \in \mathbb{A}$ we have $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ and $f(\mathcal{A}, w) \models \psi$. Therefore $\mathbb{A} \models \Diamond\psi$. On the other hand $\Box\mathbb{B} \models \neg\psi$ so for every $(\mathcal{B}, w) \in \mathbb{B}$ and every $(\mathcal{B}, v) \in \Box(\mathcal{B}, w)$ we have $(\mathcal{B}, v) \models \neg\psi$. Therefore $\mathbb{B} \models \Box\neg\psi$ and thus $\mathbb{B} \models \neg\Diamond\psi$. So the formula $\Diamond\psi$ separates the sets \mathbb{A} and \mathbb{B} and since $\text{ms}(\Diamond\psi) = \text{ms}(\psi) + 1 \leq m$ and $\text{cs}(\Diamond\psi) = \text{cs}(\psi) \leq k$, $(\text{sep})_{m,k}$ holds.
- (d) The case in which the first move of the winning strategy of S is a right successor move is similar to the case of left successor move. It suffices to switch the classes \mathbb{A} and \mathbb{B} , and replace \Diamond with \Box .

Now assume $(\text{sep})_{m,k}$ holds, and φ is the formula separating \mathbb{A} and \mathbb{B} . We obtain a winning strategy of S for the game $\text{FS}_{m,k}(\mathbb{A}, \mathbb{B})$ using φ as follows:

- (a) If φ is a literal, S wins the game with no moves.
- (b) Assume that $\varphi = \psi \vee \vartheta$. Let $\mathbb{A}_1 := \{(\mathcal{A}, w) \in \mathbb{A} \mid (\mathcal{A}, w) \models \psi\}$ and $\mathbb{A}_2 := \{(\mathcal{A}, w) \in \mathbb{A} \mid (\mathcal{A}, w) \models \vartheta\}$. Since $\mathbb{A} \models \varphi$ we have $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. In addition, since $\mathbb{B} \models \neg\varphi$, we have $\mathbb{B} \models \neg\psi$ and $\mathbb{B} \models \neg\vartheta$. Thus ψ separates the sets \mathbb{A}_1 and \mathbb{B} and ϑ separates the sets \mathbb{A}_2 and \mathbb{B} . Since $\text{ms}(\psi) + \text{ms}(\vartheta) = \text{ms}(\varphi) \leq m$, there are $m_1, m_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$, $\text{ms}(\psi) \leq m_1$ and $\text{ms}(\vartheta) \leq m_2$. Similarly since $\text{cs}(\psi) + \text{cs}(\vartheta) + 1 = \text{cs}(\varphi) \leq k$, there are $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 + 1 = k$, $\text{cs}(\psi) \leq k_1$ and $\text{cs}(\vartheta) \leq k_2$. By induction hypothesis S has winning strategies for the games $\text{FS}_{m_1, k_1}(\mathbb{A}_1, \mathbb{B})$ and $\text{FS}_{m_2, k_2}(\mathbb{A}_2, \mathbb{B})$. Since $k \geq \text{cs}(\varphi) \geq 1$, S can start the game $\text{FS}_{m, k}(\mathbb{A}, \mathbb{B})$ with a left splitting move choosing the numbers m_1, m_2, k_1 and k_2 and the sets \mathbb{A}_1 and \mathbb{A}_2 . Then S wins the game by following the winning strategy for whichever position D chooses.
- (c) The case $\varphi = \psi \wedge \vartheta$ is similar to the case of disjunction. This time S uses the sets $\mathbb{B}_1 := \{(\mathcal{B}, w) \in \mathbb{B} \mid (\mathcal{B}, w) \models \neg\psi\}$ and $\mathbb{B}_2 := \{(\mathcal{B}, w) \in \mathbb{B} \mid (\mathcal{B}, w) \models \neg\vartheta\}$ for choosing a right splitting move.
- (d) Assume that $\varphi = \Diamond\psi$. Since $\mathbb{A} \models \varphi$, for every $(\mathcal{A}, w) \in \mathbb{A}$ there is $(\mathcal{A}, v_w) \in \Box(\mathcal{A}, w)$ such that $(\mathcal{A}, v_w) \models \psi$. We define the function $f : \mathbb{A} \rightarrow \Box\mathbb{A}$ by $f(\mathcal{A}, w) = (\mathcal{A}, v_w)$. Clearly $\Diamond_f\mathbb{A} \models \psi$. On the other hand $\mathbb{B} \models \neg\varphi$ so $\mathbb{B} \models \Box\neg\psi$ and thus for each $(\mathcal{B}, w) \in \mathbb{B}$ and each $(\mathcal{B}, v) \in \Box(\mathcal{B}, w)$ we have $(\mathcal{B}, v) \models \neg\psi$. Therefore $\Box\mathbb{B} \models \neg\psi$ and the formula ψ separates the sets $\Diamond_f\mathbb{A}$ and $\Box\mathbb{B}$. Moreover, $\text{ms}(\psi) = \text{ms}(\varphi) - 1 \leq m - 1$ and $\text{cs}(\psi) = \text{cs}(\varphi) \leq k$ so by induction hypothesis S has a winning strategy for the game $\text{FS}_{m-1, k}(\Diamond_f\mathbb{A}, \Box\mathbb{B})$. Since $m \geq \text{ms}(\varphi) \geq 1$, S can start the game $\text{FS}_{m, k}(\mathbb{A}, \mathbb{B})$ with a left successor move choosing the function f . Then S wins the game by following the winning strategy for the game $\text{FS}_{m-1, k}(\Diamond_f\mathbb{A}, \Box\mathbb{B})$.
- (e) The case $\varphi = \Box\psi$ is proved in the same way as the case with \Diamond . Again it suffices to switch \mathbb{A} and \mathbb{B} , and use right successor move instead of left successor move.

□

We prove next that m -bisimilarity implies that D has winning strategy in the formula size game with modal parameter m .

Theorem 3.3 *Let \mathbb{A} and \mathbb{B} be sets of pointed models and let $m, k \in \mathbb{N}$. If there are m -bisimilar pointed models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$, then D has a winning strategy for the game $\text{FS}_{m, k}(\mathbb{A}, \mathbb{B})$.*

Proof. The proof proceeds by induction on the number $m+k \in \mathbb{N}$. If $m+k = 0$ and $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$ are m -bisimilar, then they are 0-bisimilar and thus satisfy the same literals. Thus there is no literal $\varphi \in \text{ML}$ that separates the sets \mathbb{A} and \mathbb{B} . Since S cannot make any moves and S does not win the game in this position, D wins the game $\text{FS}_{0, 0}(\mathbb{A}, \mathbb{B})$.

Assume that $m+k > 0$ and $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$ are m -bisimilar. As in the basic step, S does not win the game in this position. We consider the

cases of the first move of S in the game $\text{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.

If S starts with a left splitting move choosing the numbers m_1, m_2, k_1 and k_2 and the sets \mathbb{A}_1 and \mathbb{A}_2 , then since $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$, D can choose the next position $(m_i, k_i, \mathbb{A}_i, \mathbb{B})$, $i \in \{1, 2\}$ in such a way that $(\mathcal{A}, w) \in \mathbb{A}_i$. Then we have $m_i \leq m$ and $m_i + k_i < m + k$ so by induction hypothesis D has a winning strategy for the game $\text{FS}_{m_i, k_i}(\mathbb{A}_i, \mathbb{B})$. The case of a right splitting move is similar.

If S starts with a left successor move choosing a function $f : \mathbb{A} \rightarrow \Box \mathbb{A}$, then since (\mathcal{A}, w) and (\mathcal{B}, v) are m -bisimilar, there is a pointed model $(\mathcal{B}, v') \in \Box(\mathcal{B}, v)$ that is $m-1$ -bisimilar with the pointed model $f(\mathcal{A}, w)$. Since $m-1+k < m+k$, by induction hypothesis D has a winning strategy in $\text{FS}_{m-1, k}(\Diamond_f \mathbb{A}, \Box \mathbb{B})$. The case of a right successor move is similar. \square

4 Succinctness of FO over ML

In this section, we illustrate the use of the formula size game $\text{FS}_{m,k}$ by proving a nonelementary succinctness gap between bisimulation invariant first-order logic and modal logic.

4.1 A property of pointed frames

For the remainder of this paper we consider only the case where the set Φ of propositional symbols is empty. This makes all points in Kripke-models propositionally equivalent so we call pointed models in this section pointed frames. The only formulas available for the win condition of S in the game $\text{FS}_{m,k}$ are \perp and \top . Thus S only wins the game from the position $(m, k, \mathbb{A}, \mathbb{B})$ if either $\mathbb{A} = \emptyset$ and $\mathbb{B} \neq \emptyset$ or $\mathbb{A} \neq \emptyset$ and $\mathbb{B} = \emptyset$.

We will use the following two classes in our application of the formula size game $\text{FS}_{m,k}$:

- \mathbb{A}_n is the class of all pointed frames (\mathcal{A}, w) such that for all $(\mathcal{A}, u), (\mathcal{A}, v) \in \Box(\mathcal{A}, w)$, the frames (\mathcal{A}, u) and (\mathcal{A}, v) are n -bisimilar.
- \mathbb{B}_n is the complement of \mathbb{A}_n .

Lemma 4.1 *For each $n \in \mathbb{N}$ there is a formula $\varphi_n(x) \in \text{FO}$ that separates the classes \mathbb{A}_n and \mathbb{B}_n such that the size of $\varphi_n(x)$ is exponential with respect to n , i.e., $s(\varphi_n) = \mathcal{O}(2^n)$.*

Proof. We first define formulas $\psi_n(x, y) \in \text{FO}$ such that $(\mathcal{M}, u) \rightleftharpoons_n (\mathcal{M}, v)$ if and only if $\mathcal{M} \models \psi_n[u/x, v/y]$. The formulas $\psi_n(x, y)$ are defined recursively as follows:

$$\begin{aligned} \psi_1(x, y) &:= \exists s R(x, s) \leftrightarrow \exists t R(y, t) \\ \psi_{n+1}(x, y) &:= \forall s (R(x, s) \rightarrow \exists t (R(y, t) \wedge \psi_n(s, t)) \\ &\quad \wedge \forall t (R(y, t) \rightarrow \exists s (R(x, s) \wedge \psi_n(s, t))). \end{aligned}$$

Clearly these formulas express n -bisimilarity as intended. When we interpret the equivalences and implications as shorthand in the standard way, we get the sizes $s(\psi_1) = 11$ and $s(\psi_{n+1}) = 2 \cdot s(\psi_n) + 13$. Thus $s(\psi_n) = 3 \cdot 2^{n+2} - 13$.

Now we can define the formulas φ_n :

$$\varphi_n(x) := \forall y \forall z (R(x, y) \wedge R(x, z) \rightarrow \psi_n(y, z)).$$

Clearly for every $(\mathcal{A}, w) \in \mathbb{A}_n$ we have $\mathcal{A} \models \varphi_n[w/x]$ and for every $(\mathcal{B}, v) \in \mathbb{B}_n$ we have $\mathcal{B} \models \neg \varphi_n[v/x]$ so the formula φ_n separates the classes \mathbb{A}_n and \mathbb{B}_n . Furthermore, $s(\varphi_n) = s(\psi_n) + 6 = 3 \cdot 2^{n+2} - 7$ so the size of φ_n is exponential with respect to n . \square

Lemma 4.2 *For each $n \in \mathbb{N}$, the formula φ_n is $n + 1$ -bisimulation invariant.*

Proof. Let (\mathcal{A}, w) and (\mathcal{B}, v) be $n + 1$ -bisimilar pointed models. Assume that $\mathcal{A} \models \varphi_n[w/x]$. If $(\mathcal{B}, v_1), (\mathcal{B}, v_2) \in \square(\mathcal{B}, v)$, by $n + 1$ -bisimilarity there are $(\mathcal{A}, w_1), (\mathcal{A}, w_2) \in \square(\mathcal{A}, w)$ such that $(\mathcal{A}, w_1) \simeq_n (\mathcal{B}, v_1)$ and $(\mathcal{A}, w_2) \simeq_n (\mathcal{B}, v_2)$. Since $\mathcal{A} \models \varphi_n[w/x]$, we have $(\mathcal{B}, v_1) \simeq_n (\mathcal{A}, w_1) \simeq_n (\mathcal{A}, w_2) \simeq_n (\mathcal{B}, v_2)$ so $\mathcal{B} \models \psi_n[v_1/x, v_2/y]$. Thus, we see that $\mathcal{B} \models \varphi_n[v/x]$. \square

It follows now from van Benthem's characterization theorem that each φ_n is equivalent to some ML-formula. Thus, we get the following corollary.

Corollary 4.3 *For each $n \in \mathbb{N}$, there is a formula $\vartheta_n \in \text{ML}$ that separates the classes \mathbb{A}_n and \mathbb{B}_n .*

4.2 Set theoretic construction of pointed frames

We have shown that the classes \mathbb{A}_n and \mathbb{B}_n can be separated both in ML and in FO. Furthermore the size of the FO-formula is exponential with respect to n . It only remains to ask: what is the size of the smallest ML-formula that separates the classes \mathbb{A}_n and \mathbb{B}_n ? To answer this we will need suitable subsets of \mathbb{A}_n and \mathbb{B}_n to play the formula size game on.

Definition 4.4 Let $n \in \mathbb{N}$. The finite levels of the cumulative hierarchy are defined recursively as follows:

$$\begin{aligned} V_0 &= \emptyset \\ V_{n+1} &= \mathcal{P}(V_n) \end{aligned}$$

For every $n \in \mathbb{N}$, V_n is a *transitive set*, i.e., for every $a \in V_n$ and every $b \in a$ it holds that $b \in V_n$. Thus it is reasonable to define a frame $\mathcal{F}_n = (V_n, R_n)$, where for all $a, b \in V_n$ it holds that $(a, b) \in R_n \Leftrightarrow b \in a$.

For every set $a \in V_n$ we define a pointed frame (\mathcal{M}_a, a) , where \mathcal{M}_a is the subframe of \mathcal{F}_n generated by the point a .

Lemma 4.5 *Let $n \in \mathbb{N}$ and $a, b \in V_{n+1}$. If $a \neq b$, then $(\mathcal{M}_a, a) \not\simeq_n (\mathcal{M}_b, b)$.*

Proof. We prove the claim by induction on n . The basic step $n = 0$ is trivial since V_1 only has one element. For the induction step, assume that $a, b \in V_{n+1}$ and $a \neq b$. Assume further for contradiction that $(\mathcal{M}_a, a) \simeq_n (\mathcal{M}_b, b)$. Since $a \neq b$, by symmetry we can assume that there is $x \in a$ such that $x \notin b$. By n -bisimilarity there is $y \in b$ such that (\mathcal{M}_x, x) and (\mathcal{M}_y, y) are $n - 1$ -bisimilar. Since $x \in a \in V_{n+1}$ and $y \in b \in V_{n+1}$, we have $x, y \in V_n$. By induction hypothesis we obtain $x = y$. This is a contradiction, since $x \notin b$ and $y \in b$. \square

Let \mathbb{A} be a set of pointed frames and $w \notin \text{dom}(\mathcal{A})$ for all $(\mathcal{A}, v) \in \mathbb{A}$. We use the notation $\Delta\mathbb{A} := (\mathcal{M}, w)$, where

$$\begin{aligned}\text{dom}(\mathcal{M}) &= \{w\} \cup \bigcup \{\text{dom}(\mathcal{A}) \mid (\mathcal{A}, v) \in \mathbb{A}\}, \text{ and} \\ R^{\mathcal{M}} &= \{(w, v) \mid (\mathcal{A}, v) \in \mathbb{A}\} \cup \bigcup \{R^{\mathcal{A}} \mid (\mathcal{A}, v) \in \mathbb{A}\}.\end{aligned}$$

For $n \in \mathbb{N}$ we further use the notation

$$\begin{aligned}\mathbb{V}_n &:= \{\Delta\{(\mathcal{M}_a, a)\} \mid a \in V_{n+1}\} \\ \mathbb{E}_n &:= \{\Delta\{(\mathcal{M}_a, a), (\mathcal{M}_b, b)\} \mid a, b \in V_{n+1}, a \neq b\}\end{aligned}$$

It is well known that the cardinality of V_n is the exponential tower of $n - 1$. Thus, the cardinality of \mathbb{V}_n is $\text{twr}(n)$.

Lemma 4.6 *If $n \in \mathbb{N}$, we have $|\mathbb{V}_n| = |V_{n+1}| = \text{twr}(n)$.* \square

4.3 Graph colorings and winning strategies in $\text{FS}_{m,k}$

Our aim is to prove that any ML-formula ϑ_n separating the sets \mathbb{V}_n and \mathbb{E}_n is of size at least $\text{twr}(n - 1)$. To do this, we make use of a surprising connection between the chromatic numbers of certain graphs related to pairs of the form (\mathbb{V}, \mathbb{E}) , and existence of a winning strategies for D in the game $\text{FS}_{m,k}(\mathbb{V}, \mathbb{E})$.

Let $n \in \mathbb{N}$, $\emptyset \neq \mathbb{V} \subseteq \mathbb{V}_n$ and $\mathbb{E} \subseteq \mathbb{E}_n$. Then $\mathcal{G}(\mathbb{V}, \mathbb{E})$ denotes the graph (V, E) , where

$$\begin{aligned}V &= \{(\mathcal{M}, w) \mid \Delta\{(\mathcal{M}, w)\} \in \mathbb{V}\}, \text{ and} \\ E &= \{((\mathcal{M}, w), (\mathcal{M}', w')) \in V \times V \mid \Delta\{(\mathcal{M}, w), (\mathcal{M}', w')\} \in \mathbb{E}\}.\end{aligned}$$

Definition 4.7 Let $\mathcal{G} = (V, E)$ be a graph and let C be a set. A function $\chi : V \rightarrow C$ is a *coloring* of the graph \mathcal{G} if for all $u, v \in V$ it holds that if $(u, v) \in E$, then $\chi(u) \neq \chi(v)$. If the set C has k elements, then χ is called a *k-coloring* of \mathcal{G} .

The *chromatic number* of \mathcal{G} , denoted by $\chi(\mathcal{G})$, is the smallest number $k \in \mathbb{N}$ for which there is a k -coloring of \mathcal{G} .

Lemma 4.8 *Let $\mathcal{G} = (V, E)$ be a graph.*

- (i) *Let $V_1, V_2 \subseteq V$ be nonempty such that $V_1 \cup V_2 = V$ and let $\mathcal{G}_1 = (V_1, E \upharpoonright V_1)$ and $\mathcal{G}_2 = (V_2, E \upharpoonright V_2)$. Then we have $\chi(\mathcal{G}) \leq \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2)$.*
- (ii) *Let $E_1, E_2 \subseteq E$ such that $E_1 \cup E_2 = E$ and let $\mathcal{G}_1 = (V, E_1)$ and $\mathcal{G}_2 = (V, E_2)$. Then $\chi(\mathcal{G}) \leq \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$.*

Proof. (i) Let V_1, V_2, \mathcal{G}_1 and \mathcal{G}_2 be as in the claim and let $k_1 = \chi(\mathcal{G}_1)$ and $k_2 = \chi(\mathcal{G}_2)$. Let $\chi_1 : V_1 \rightarrow \{1, \dots, k_1\}$ be a k_1 -coloring of the graph \mathcal{G}_1 and let $\chi_2 : V_2 \rightarrow \{k_1 + 1, \dots, k_1 + k_2\}$ be a k_2 -coloring of the graph \mathcal{G}_2 . Then it is straightforward to show that $\chi = \chi_1 \cup (\chi_2 \upharpoonright (V_2 \setminus V_1))$ is a $k_1 + k_2$ -coloring of the graph \mathcal{G} , whence $\chi(\mathcal{G}) \leq k_1 + k_2 = \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2)$.

(ii) Let $\chi_1 : V \rightarrow \{1, \dots, k_1\}$ and $\chi_2 : V \rightarrow \{1, \dots, k_2\}$ be colorings of the graphs \mathcal{G}_1 and \mathcal{G}_2 , respectively. Then it is easy to verify that the map

$\chi : V \rightarrow \{1, \dots, k_1\} \times \{1, \dots, k_2\}$ defined by $\chi(v) = (\chi_1(v), \chi_2(v))$ is a coloring of \mathcal{G} . Thus we obtain $\chi(\mathcal{G}) \leq |\{1, \dots, k_1\} \times \{1, \dots, k_2\}| = \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$. \square

Lemma 4.9 *Assume $\emptyset \neq \mathbb{V} \subseteq \mathbb{V}_n$ and $\mathbb{E} \subseteq \mathbb{E}_n$ for some $n \in \mathbb{N}$ and let $m, k \in \mathbb{N}$. If $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$ and $k < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})))$, then D has a winning strategy in the game $\text{FS}_{m,k}(\mathbb{V}, \mathbb{E})$.*

Proof. Let $n, m, k \in \mathbb{N}$ and assume that $\emptyset \neq \mathbb{V} \subseteq \mathbb{V}_n$, $\mathbb{E} \subseteq \mathbb{E}_n$, $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$ and $k < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})))$. We prove the claim by induction on k .

If $k = 0$, S can only make successor moves. Since $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$, there are $(\mathcal{M}, w), (\mathcal{M}', w') \in V$ such that $((\mathcal{M}, w), (\mathcal{M}', w')) \in E$. Thus $\Delta\{(\mathcal{M}, w)\}, \Delta\{(\mathcal{M}', w')\} \in \mathbb{V}$ and $\Delta\{(\mathcal{M}, w), (\mathcal{M}', w')\} \in \mathbb{E}$. If S makes a left or right successor move, then in the resulting position $(m-1, 0, \mathbb{V}', \mathbb{E}')$ it holds that $(\mathcal{M}, w) \in \mathbb{V}' \cap \mathbb{E}'$ or $(\mathcal{M}', w') \in \mathbb{V}' \cap \mathbb{E}'$. Thus the same pointed model is present on both sides of the game and by Theorem 3.3, D has a winning strategy for the game $\text{FS}_{m,k}(\mathbb{V}', \mathbb{E}')$.

Assume then that $k > 0$. If S starts the game with a successor move, then D wins as described above.

Assume that S begins the game with a left splitting move choosing the numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ and the sets $\mathbb{V}_1, \mathbb{V}_2 \subseteq \mathbb{V}$. Consider the graphs $\mathcal{G}(\mathbb{V}, \mathbb{E}) = (V, E)$, $\mathcal{G}(\mathbb{V}_1, \mathbb{E}) = (V_1, E_1)$ and $\mathcal{G}(\mathbb{V}_2, \mathbb{E}) = (V_2, E_2)$. Since $\mathbb{V}_1 \cup \mathbb{V}_2 = \mathbb{V}$, we have $V_1 \cup V_2 = V$. In addition, by the definition of the graphs $\mathcal{G}(\mathbb{V}, \mathbb{E})$, $\mathcal{G}(\mathbb{V}_1, \mathbb{E})$ and $\mathcal{G}(\mathbb{V}_2, \mathbb{E})$ we see that $E_1 = E \upharpoonright V_1$ and $E_2 = E \upharpoonright V_2$. Thus by Lemma 4.8, we obtain $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \leq \chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})) + \chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E}))$. It must hold that $k_1 < \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})))$ or $k_2 < \log(\chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E})))$, since otherwise we would have

$$\begin{aligned} k &< \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}))) \leq \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})) + \chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E}))) \\ &\leq \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E}))) + \log(\chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E}))) + 1 \leq k_1 + k_2 + 1 = k. \end{aligned}$$

Thus D can choose the next position of the game, $(m_i, k_i, \mathbb{V}_i, \mathbb{E})$, in such a way that $k_i < \log(\chi(\mathcal{G}(\mathbb{V}_i, \mathbb{E})))$. By induction hypothesis D has a winning strategy in the game $\text{FS}_{m_i, k_i}(\mathbb{V}_i, \mathbb{E})$.

Assume then that S begins the game with a right splitting move choosing the numbers $m_1, m_2, k_1, k_2 \in \mathbb{N}$ and the sets $\mathbb{E}_1, \mathbb{E}_2 \subseteq \mathbb{E}$. Consider now the graphs $\mathcal{G}(\mathbb{V}, \mathbb{E}) = (V, E)$, $\mathcal{G}(\mathbb{V}, \mathbb{E}_1) = (V_1, E_1)$ and $\mathcal{G}(\mathbb{V}, \mathbb{E}_2) = (V, E_2)$. Clearly $V_1 = V_2 = V$ and since $\mathbb{E}_1 \cup \mathbb{E}_2 = \mathbb{E}$, we have $E_1 \cup E_2 = E$. Thus by Lemma 4.8, we obtain $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \leq \chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1))\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2))$. It must hold that $k_1 < \log(\chi(\mathcal{G}(\mathbb{V}_1, \mathbb{E})))$ or $k_2 < \log(\chi(\mathcal{G}(\mathbb{V}_2, \mathbb{E})))$, since otherwise we would have

$$\begin{aligned} k &< \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}))) \leq \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1))\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2))) \\ &= \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_1))) + \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E}_2))) \leq k_1 + k_2 + 1 = k. \end{aligned}$$

Thus D can again choose the next position of the game, $(m_i, k_i, \mathbb{V}, \mathbb{E}_i)$, in such a way that $k_i < \log(\chi(\mathcal{G}(\mathbb{V}_i, \mathbb{E})))$. By induction hypothesis D has a winning strategy in the game $\text{FS}_{m_i, k_i}(\mathbb{V}, \mathbb{E}_i)$. \square

Lemma 4.10 *If $k < \text{twr}(n - 1)$ and $m \in \mathbb{N}$, then D has a winning strategy in the game $\text{FS}_{m,k}(\mathbb{V}_n, \mathbb{E}_n)$.*

Proof. By Lemma 4.6, we have $|\mathbb{V}_n| = \text{twr}(n)$ and the set \mathbb{E}_n consists of all the pointed frames $\Delta\{(\mathcal{M}, w), (\mathcal{M}', w')\}$, where $(\mathcal{M}, w), (\mathcal{M}', w') \in \mathbb{V}_n$, $(\mathcal{M}, w) \neq (\mathcal{M}', w')$. Thus the graph $\mathcal{G}(\mathbb{V}_n, \mathbb{E}_n)$ is isomorphic with the complete graph $K_{\text{twr}(n)}$. Therefore we obtain

$$\chi(\mathcal{G}(\mathbb{V}_n, \mathbb{E}_n)) = \chi(K_{\text{twr}(n)}) = \text{twr}(n).$$

By the assumption, $k < \text{twr}(n - 1) = \log(\text{twr}(n)) = \log(\chi(\mathcal{G}(\mathbb{V}_n, \mathbb{E}_n)))$, so by Lemma 4.9, D has a winning strategy in the game $\text{FS}_{m,k}(\mathbb{V}_n, \mathbb{E}_n)$. \square

Theorem 4.11 *Let $n \in \mathbb{N}$. If a formula $\vartheta_n \in \text{ML}$ separates the classes \mathbb{A}_n and \mathbb{B}_n , then $s(\vartheta_n) \geq \text{twr}(n - 1)$.*

Proof. Assume that a formula $\vartheta_n \in \text{ML}$ separates the classes \mathbb{A}_n and \mathbb{B}_n . Since for every pointed frame $(M, w) \in \mathbb{V}_n$ the set $\Box(\mathcal{M}, w)$ has only one element, we have $\mathbb{V}_n \subseteq \mathbb{A}_n$. On the other hand, every pointed frame in the set \mathbb{E}_n is of the form $\Delta\{(\mathcal{M}_a, a), (\mathcal{M}_b, b)\}$, where $a, b \in V_{n+1}$, $a \neq b$. By Lemma 4.5, $(\mathcal{M}_a, a) \not\equiv_n (\mathcal{M}_b, b)$ so $\mathbb{E}_n \subseteq \mathbb{B}_n$.

Assume for contradiction that $s(\vartheta_n) < \text{twr}(n - 1)$. By Theorem 3.2, S has a winning strategy in the game $\text{FS}_{m,k}(\mathbb{V}_n, \mathbb{E}_n)$ for $m = \text{ms}(\vartheta_n)$ and $k = \text{cs}(\vartheta_n)$. On the other hand, $k < \text{twr}(n - 1)$, whence by Lemma 4.10, D has a winning strategy in the same game. \square

We now have everything we need for proving the nonelementary succinctness of FO over ML. By Lemma 4.1, for each $n \in \mathbb{N}$ there is a formula $\varphi_n(x) \in \text{FO}$ such that φ_n separates the classes \mathbb{A}_n and \mathbb{B}_n with $s(\varphi) = \mathcal{O}(2^n)$. On the other hand by Corollary 4.3, there is an equivalent formula $\vartheta_n \in \text{ML}$, but by Theorem 4.11 the size of ϑ_n must be at least $\text{twr}(n - 1)$.

Corollary 4.12 *Bisimulation invariant FO is nonelementarily more succinct than ML.*

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